AP Physics C. Eddy Currents.

> A very strong magnet (e.g., a neodymium-iron-boron alloy) dropped down a vertical copper pipe does not fall with constant acceleration, but instead glides down at a constant velocity; the gravitational force is quickly met by an equal and opposite magnetic force upwards. Find the terminal velocity.

It is pretty clear that this magnetic force arises from Faraday's Law. Here is what is happening: as the magnet drops, it induces *eddy currents* in the pipe, and these in turn create a non-uniform magnetic field which acts on the dipole. A full explanation of eddy currents is difficult; however, the main ideas are accessible to anyone who has had a calculus-based electromagnetism course. Eddy currents also explain why refrigerator magnets stick to refrigerators. Oddly, the subject of eddy currents is almost completely ignored in most textbooks. Most physicists of the past two generations are not very familiar with them, even though the great James Clerk Maxwell himself explained how they worked back in 1872. Most of what follows I learned from a series of terrific papers by W. M. Saslow (cited at the end.)

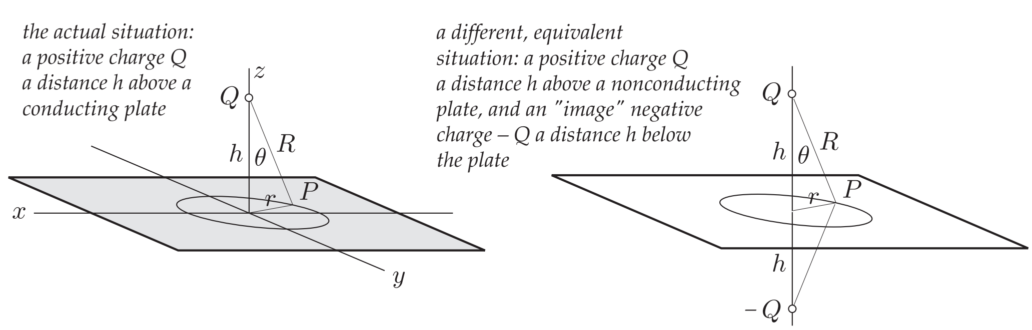
1. The method of images (from E. M. Purcell, *Electricity and Magnetism*, first ed., p. 92)

Eventually we are going to look at a dipole magnet above a superconducting sheet. Electrical fields are easier, so let's start with that.

> A point charge Q not too far above a very large conducting plate induces negative charges on the plate. Find the distribution of these negative charges on the plate, i.e. find $\sigma(r)$, where r is the distance from the z-axis. (We know it is going to be symmetric with respect to the z-axis.)

For simplicity, let the plate lie in the x-y plane, and let the charge be at a point $(0, 0, h)$ on the z-axis. Because the conducting plate has to be an equipotential, the electric field lines starting on the positive charge Q have to end up on the plate at right angles to the plate. (Remember, the electric field lines have to be perpendicular to a line or a surface of constant φ.)

What we can do is find a simpler problem that has exactly the same solution on the plane. Imagine, instead of the problem we have, a related problem of two charges, one $+Q$ and the other $-Q$, located at $(0, 0, h)$ and $(0, 0, -h)$ respectively. It is easy to see by symmetry that the field lines will be perpendicular to the x-y plane, as required, and begin at the $+Q$ charge as required. It is true that there will be a field below the x-y plane with the two charges, and not with the original problem, but we will just ignore the solution for $z < 0$, instead choosing $E = 0$ for $z < 0$. This second problem imagines an "image" charge of $-Q$ below a non-conducting plane to provide the exact same electric field as that due to the induced negative charge on the conducting plane. If you imagine the plane as a mirror, you can see why the negative charge $-Q$ is called an "image". The inventor of this technique seems to have been William Thomson, or Lord Kelvin as he is usually known.



Can we find the electric field at the plane? Certainly. Take a point (x, y) on the plane, and the z component of the electric field due to both charges is given by

$$E_z = -\frac{kQ}{R^2}\cos\theta - \frac{kQ}{R^2}\cos\theta = -\frac{2kQ}{R^2}\cos\theta$$

where $\cos\theta = \dfrac{h}{R}$, and $R = \sqrt{r^2 + h^2}$. That is,

$$E_z = -\frac{2kQh}{R^3} = -\frac{2kQh}{(r^2 + h^2)^{3/2}}$$

Recall that for a conducting plane, we have a simple connection between the electric field E_z and the charge density, σ;

$$E_z = 4\pi k\sigma$$

which means that $\sigma = E_z/4\pi k$; in this case,

$$\sigma(r) = \frac{-Qh}{2\pi(r^2 + h^2)^{3/2}}$$

This is our answer. We can check it if we find the total charge on the plane; it *should* be equal to $-Q$. Let's see:

$$\begin{aligned}\text{total surface charge} &= \int \sigma\, dA = \int_0^\infty \sigma\, 2\pi r\, dr \\ &= -\tfrac{1}{2}Qh \int_0^\infty \frac{2r}{(r^2 + h^2)^{3/2}}\, dr\end{aligned}$$

Perform the integral with an easy u substitution; let $u = r^2 + h^2$. Then when $r = 0$, $u = h^2$, and when $r = \infty$, $u = \infty$ also. The integral becomes

$$\text{total surface charge} = -\tfrac{1}{2}Qh \int_{h^2}^\infty u^{-3/2}\, du = Qhu^{-1/2}\Big|_{h^2}^\infty = -Q$$

as expected!

2. A magnet above a superconducting sheet.

In exactly the same manner of the previous problem, imagine now a dipole magnet oriented along the z axis, with south pole lower than the north pole, above a large superconducting sheet. It is an experimental fact that magnetic fields cannot penetrate superconductors (this is called the *Meissner effect*.) How does the superconductor prevent the magnetic field of the dipole from entering? Easy: it produces eddy currents within itself to exactly counter the dipole's magnetic field at its surface. We can describe this magnetic field with an image dipole (of the opposite orientation.)

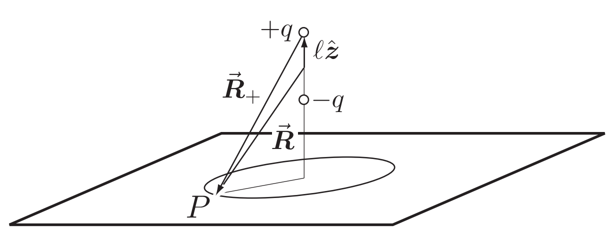
An astonishing consequence of this is that the magnet will *levitate* above the superconductor! We can use the method of images to determine the equilibrium height at which this occurs.

First, we need an expression for the field of a magnetic dipole. Get this by considering the field of an *electric* dipole, which is easy to find. Let a positive charge q be at the point $(0, 0, h + \ell)$, and a negative charge $-q$ be at the point $(0, 0, h - \ell)$. Let $\vec{\boldsymbol{R}}$ be the vector from the point $(0, 0, h)$ to the point $P(x, y, 0)$. Then the vector from the positive charge to P is

$$\vec{\boldsymbol{R}}_+ = \vec{\boldsymbol{R}} - \ell\hat{\boldsymbol{z}}$$

The potential φ_+ at point P due to the positive charge is

$$\varphi_+ = \frac{kq}{R_+} = \frac{kq}{\left|\vec{\boldsymbol{R}} - \ell\hat{\boldsymbol{z}}\right|}$$



We can use Taylor's theorem on this expression; we have

$$f(R_+) = f(R) - \ell\hat{\boldsymbol{z}} \cdot \nabla f(R) + \ldots$$

As you should be able to show,

$$\nabla f(R) = \hat{\boldsymbol{R}}\,\frac{df}{dR}$$

and so

$$\nabla \frac{1}{R} = -\frac{\hat{\boldsymbol{R}}}{R^2} = -\frac{\vec{\boldsymbol{R}}}{R^3}.$$

Consequently,

$$\varphi_+ = \frac{kq}{R} + \frac{kq\ell\hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{R}}}{R^3} + \ldots$$

and in exactly the same way,

$$\varphi_- = -\frac{kq}{R} + \frac{k(-q)\ell(-\hat{\boldsymbol{z}}) \cdot \vec{\boldsymbol{R}}}{R^3} + \ldots$$

so that

$$\varphi_{\text{dipole}} = \varphi_+ + \varphi_- = +\frac{2kq\ell\hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{R}}}{R^3}$$

where we drop the other terms (which will not be important, provided ℓ is $\ll R$.) It is traditional to introduce the *electric dipole moment*, $\vec{\boldsymbol{p}}$, as

$$\vec{\boldsymbol{p}} = 2q\ell\hat{\boldsymbol{z}} = q\vec{\boldsymbol{d}}$$

(basically, one draws the direction of the dipole moment *from the negative to the positive charge*; the "moment" is the vector $\vec{\boldsymbol{d}} = \vec{\boldsymbol{R}}_+ - \vec{\boldsymbol{R}}_-$ from the negative to the positive charge; which is $2\ell\hat{\boldsymbol{z}}$ in this case. This direction is chosen so that the dipole *lines up* with an external electric field.) Then

$$\varphi_{\text{dipole}} = \frac{k\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}}}{R^3}$$

The electric field of the dipole is given by the gradient of this potential;

$$\vec{\boldsymbol{E}}_{\text{dipole}} = -\nabla\varphi_{\text{dipole}}$$

Let's look at this in detail.

$$\nabla\left(\frac{\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}}}{R^3}\right) = \left(\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}}\right)\nabla\frac{1}{R^3} + \left(\frac{1}{R^3}\right)\nabla\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}}$$

by the product rule. The first term is found by the rule given above, and is equal to

$$\left(\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}}\right)\left(-3\frac{\hat{\boldsymbol{R}}}{R^4}\right) = -\frac{(3\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{R}})\hat{\boldsymbol{R}}}{R^3}$$

Recall that $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}} = p_x x + p_y y + p_z z$. Then $\partial(p_x x + p_y y + p_z z)/\partial x = p_x$, and similarly for the other derivatives, which gives

$$\nabla\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{R}} = \vec{\boldsymbol{p}}$$

so that

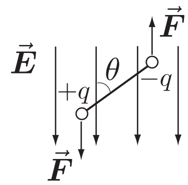
$$\vec{\boldsymbol{E}}_{\text{dipole}} = k\frac{(3\vec{\boldsymbol{p}} \cdot \hat{\boldsymbol{R}})\hat{\boldsymbol{R}} - \vec{\boldsymbol{p}}}{R^3}$$

We can simply write down the equivalent magnetic field for a magnetic dipole; introduce $\vec{\boldsymbol{m}}$ for the *magnetic* dipole moment; then

$$\vec{\boldsymbol{B}}_{\text{dipole}} = k'\frac{(3\vec{\boldsymbol{m}} \cdot \hat{\boldsymbol{R}})\hat{\boldsymbol{R}} - \vec{\boldsymbol{m}}}{R^3}$$

To determine the position of the dipole magnet above a superconducting sheet, we will write down a formula for the energy of the dipole as a function of its height h above the sheet, and then minimize the energy. First we have to find the magnetic energy of the dipole.

Again, we rely on analogy with an electric dipole. Imagine an electric dipole as before in a uniform electric field. The net *force* on the dipole will be zero, because the forces on either end are equal and opposite. There will, however, be a *torque*:



The torque will be given by

$$\vec{\boldsymbol{\tau}} = \tfrac{1}{2}\vec{\boldsymbol{d}} \times q\vec{\boldsymbol{E}} + \tfrac{1}{2}\vec{\boldsymbol{d}} \times q\vec{\boldsymbol{E}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}$$

The size of the torque is $pE\sin\theta$, and the energy involved in turning the dipole in the field is given by

$$\text{Work} = U_{\text{dipole}} = \int pE\,\sin\theta\,d\theta = -pE\,\cos\theta = -\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{E}}$$

so that we should expect that for the magnetic dipole, an energy of the form

$$U_{\text{dipole}} = -\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{B}}$$

and for a magnetic dipole in a constant magnetic field, sure enough, that is the right formula. However, things are a little more complicated in this case. The magnetic field of the superconducting plate, which we are describing by the image dipole, would not be present except for the other dipole; that is, this magnetic field is induced by the real magnet. We might imagine that there would be a linear relation between the two; say

$$B_{\text{image}} = \lambda m$$

where λ is some constant. That is, we should say

$$dU = -m\cos\theta\,dB = -\lambda m\cos\theta\,dm$$

or

$$U_{\text{dipole}} = -\tfrac{1}{2}\lambda m^2\cos\theta = -\tfrac{1}{2}\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{B}}_i$$

The magnetic field due to the image dipole is, from before,

$$\vec{\boldsymbol{B}}_i = k'\frac{(3\vec{\boldsymbol{m}}\cdot\hat{\boldsymbol{R}})\hat{\boldsymbol{R}} - \vec{\boldsymbol{m}}}{R^3}$$

Let the real magnet be oriented with its north pole along the z axis, and let it be a height h above the the superconducting plate. Then $\vec{\boldsymbol{m}}$ is oriented parallel to $\hat{\boldsymbol{z}}$, and $\vec{\boldsymbol{R}} = 2h\hat{\boldsymbol{z}}$, because the image dipole is as far below the plate as the real dipole is above. The image field has the opposite orientation (to repel the real dipole.) That means

$$\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{B}}_i = -k'\frac{3m^2 - m^2}{(2h)^3}$$

or

$$U_{\text{dipole}} = k'\frac{m^2}{(2h)^3}$$

To this we add the standard $U_{\text{grav}} = Mgh$; the magnet has a mass M. The total energy is given by

$$U = k'\frac{m^2}{(2h)^3} + Mgh$$

Minimize this by taking the derivative with respect to h and setting it equal to zero;

$$\frac{dU}{dh} = 0 = -3k'\frac{m^2}{8h^4} + Mg$$

or

$$h = \sqrt[4]{\frac{3k'm^2}{8Mg}}$$

Basically, the magnet induces surface eddy currents that act to repel (and expel) the magnet's field from penetrating the conductor, in obedience to the Meissner effect. Exactly the opposite occurs with ordinary magnets and ferromagnetic materials (for instance, the refrigerator's door); then the magnet induces eddy currents which produce the *same* orientation of magnetic field as the magnet, and hence attraction, rather than repulsion. A material which behaves this way is said to be *permeable*; the degree to which the material follows the lead of the external magnet is called its *permeability*. The superconductor is completely impermeable; ideally iron can be taken as infinitely permeable.

3. A strong magnet dropped into a conducting cylinder.

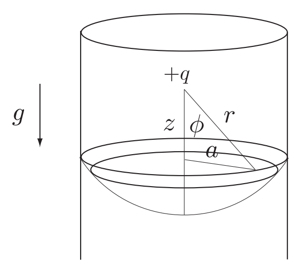
As the magnet falls, its poles induce currents in the cylinder (which we will model as a stack of circular loops of wire; each loop has an area, so the falling magnet changes flux and by Faraday's law induces a current.) These eddy currents produce a magnetic field, which then attracts the magnet, slowing the fall. The faster the fall, the more rapid the flux change, the greater the current, and the greater the resistive force, so the magnet reaches a terminal velocity quickly.

There are two ways to proceed. The more difficult is to calculate explicitly the force between the dipole and the induced magnetic field. Much easier is to calculate the energy balance; the power put into the system by gravity (force times velocity, or Mgv) must equal the power dissipated by Joule heating (current times voltage, or $\int \mathcal{E} dI$). We'll do that, rather than calculate the forces. But for completeness (and to answer, finally, Will Diamond '07's question) here is the formula for the force on a dipole:

$$F_{\text{mag on a dipole}} = (\vec{\boldsymbol{m}} \cdot \nabla)\vec{\boldsymbol{B}}$$

This formula makes sense because of the argument given earlier; a *uniform* magnetic field cannot exert a force on a dipole because the separate forces on either pole cancel. You can, if you feel virtuous, put in the expression for the magnetic field of a dipole derived above, find the gradient of it, and obtain the formula for the force between two dipoles. It ain't pretty.

So, let's balance the energies. Once again we proceed with the analogy of an electric dipole. We need to find the flux associated with either pole; the time derivative of this flux will give us the voltage that produces the eddy currents. Imagine dropping a charge into a cylinder of inner radius a. What flux will be experienced at the cylinder walls? Consider a ring of the cylinder a distance z below the charge. The electric field is spherically symmetric, so the geometry is like this:



Charge falling in a pipe:
Element of flux
$$d\Phi_E = E \times 2\pi r \sin\phi \, r d\phi$$

The integration goes from $\phi = 0$ to $\phi = \sin^{-1}(a/r)$. That is,

$$\Phi_{E_+} = E \times \int_0^{\sin^{-1}(a/r)} 2\pi r^2 \sin\phi \, d\phi = 2\pi r^2 E \times -\cos\phi \Big|_0^{\sin^{-1}(a/r)} = 2\pi r^2 E \left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

We know that the flux over the entire sphere is $E \cdot A_{\text{sphere}} = 4\pi k q$, so

$$\begin{aligned} \Phi_{E_+} &= E \cdot A_{\text{cap}} = E \cdot A_{\text{sphere}} \times \frac{A_{\text{cap}}}{A_{\text{sphere}}} \\ &= 4\pi k q \times \frac{2\pi r^2}{4\pi r^2}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) = 2\pi k q \left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \end{aligned}$$

Let the negative charge be located a distance of 2ℓ above the positive charge; and let $z = 0$ correspond to the top of the cylinder. Then

$$\Phi_{E_-} = 2\pi k(-q)\left(1 - \frac{z-2\ell}{\sqrt{a^2+(z-2\ell)^2}}\right)$$

and the net electric flux is

$$\Phi_E = 2\pi kq\left(\frac{z-2\ell}{\sqrt{a^2+(z-2\ell)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Let the magnetic dipole be regarded as two magnetic monopoles, q^* and $-q^*$, separated by a distance 2ℓ; ultimately we will say $m = 2q^*\ell$ is the magnetic moment of the dipole. (There aren't any magnetic monopoles, but we can pretend, and so by analogy find what we need.) Then the net *magnetic* flux should be

$$\Phi_B = 2\pi k' q^*\left(\frac{z-2\ell}{\sqrt{a^2+(z-2\ell)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Faraday's Law implies

$$\begin{aligned}\mathcal{E} = -\frac{d\Phi_B}{dt} &= 2\pi k' q^*\left(\frac{1}{\sqrt{a^2+(z-2\ell)^2}} - \frac{1}{\sqrt{a^2+z^2}} - \frac{(z-2\ell)^2}{(a^2+(z-2\ell)^2)^{3/2}} + \frac{z^2}{(a^2+z^2)^{3/2}}\right)\frac{dz}{dt}\\ &= 2\pi k' q^* a^2 v\left(\frac{1}{(a^2+(z-2\ell)^2)^{3/2}} - \frac{1}{(a^2+z^2)^{3/2}}\right)\end{aligned}$$

The power dissipated by a single ring of width dz is $dP = \mathcal{E}\,dI$. Let the conductance $G = 1/R$, so that $I = \mathcal{E}G$, and $dI = \mathcal{E}\,dG$. Recall the rule about resistance, R; $R = \rho L/A$. Then $dG = (1/\rho L)dA$. The ring has a length $L = 2\pi a$, and a cross-sectional area $w \times dz$. That gives

$$dP = \mathcal{E}\,dI = \mathcal{E}^2\frac{w}{2\pi a\rho}\,dz = \frac{4\pi^2 k'^2 q^{*2} a^4 v^2 w}{2\pi a\rho}\left[\frac{1}{(a^2+(z-2\ell)^2)^{3/2}} - \frac{1}{(a^2+z^2)^{3/2}}\right]^2 dz$$

The integration is to be conducted over the length of the tube. There is no reason not to imagine the tube infinite; the magnetic field dies off pretty quickly with distance, and the magnet reaches terminal velocity quickly. (Also, this makes the integration easier.) Then

$$P = \frac{2\pi k'^2 q^{*2} a^3 v^2 w}{\rho}\int_{-\infty}^{\infty}\left[\frac{1}{(a^2+(z-2\ell)^2)^{3/2}} - \frac{1}{(a^2+z^2)^{3/2}}\right]^2 dz$$

The integral can be converted into something more tractable. First, rescale by letting $z = ay$, and $2\ell = ax$. Then the integral becomes

$$\frac{1}{a^5}\int_{-\infty}^{\infty}\left[\frac{1}{(1+(y-x)^2)^{3/2}} - \frac{1}{(1+y^2)^{3/2}}\right]^2 dy = \frac{1}{a^5}f(x)$$

so that

$$P = \frac{2\pi k'^2 q^{*2} v^2 w}{\rho a^2}f(x)$$

The function $f(x)$ never gets larger than about 2.356. For small x, $f(x)$ is approximately parabolic. This can be seen by using Taylor's theorem;

$$\frac{1}{(1+(y-x)^2)^{3/2}} \approx \frac{1}{(1+y^2)^{3/2}} + \frac{3xy}{(1+(y)^2)^{5/2}}$$

so

$$\left[\frac{1}{(1+(y-x)^2)^{3/2}} - \frac{1}{(1+y^2)^{3/2}}\right]^2 \approx \frac{9x^2y^2}{(1+y^2)^5}$$

and plugging this into the integral gives

$$f(x) = 9x^2 \int_{-\infty}^{\infty} \frac{y^2}{(1+y^2)^5}\, dy = 9x^2 \times \frac{5\pi}{128} = \frac{45\pi}{128} x^2$$

(The integral can be looked up on line or in a table of integrals, or may be worked out with a couple of techniques. Consider for example differentiating $\int \frac{1}{a^2+y^2}\, dy = \pi/a$ with respect to a, and repeat.) This gives

$$P = \frac{2\pi k'^2 q^{*2} v^2 w}{\rho a^2} \frac{45\pi}{128} \left(\frac{2\ell}{a}\right)^2 = \frac{45\pi^2 k'^2 (2\ell q^*)^2 v^2 w}{64\rho a^4}$$

Recall that $2\ell q^* = m$, the magnetic dipole moment. Also, we have $P = Mgv$ from before; so setting these equal to each other gives

$$\frac{45\pi^2 k'^2 m^2 v^2 w}{64\rho a^4} = Mgv$$

and solving, finally, for v gives

$$\boxed{v = \frac{64 M g \rho a^4}{45\pi^2 k'^2 m^2 w}}$$

Does this make sense physically? As Mg increases, v increases; that's reasonable; a greater downward pull would produce a greater terminal velocity. As ρ increases, v increases: this is also reasonable, because increased ρ produces smaller current, hence less resistance to the fall (as there is less current for the magnetic field to grab on to.) By the same argument, as w, the thickness of the pipe, increases, the resistance decreases, hence greater current, hence greater magnetic force resisting, hence lesser v. Also, as the magnetic moment m increases, the speed decreases, as expected. The fourth power dependence of the radius of the pipe is a little unexpected; surely the radius matters (as it increases the resistance of the individual coils) and also, the farther away the coils are, the less the magnetic field can influence the currents. So the radius is hugely important.

Experimentally it turns out that the approximation of $f(x)$ as parabolic does not hold up, and produces about a 5% error. But the main features of the phenomenon seem to be explained.

One may also obtain the expression for v by balancing forces. This is shown in the second of Saslow's papers referenced below.

References

E. M. Purcell, *Electricity and Magnetism*, Berkeley Physics Course, McGraw-Hill 1965.

W. M. Saslow, "How a superconductor supports a magnet, how magnetically 'soft' iron attracts a magnet, and eddy currents for the uninitiated", *American Journal of Physics*, **59** 16-25.

W. M. Saslow, "Maxwell's theory of eddy currents in thin conducting sheets, and applications to electromagnetic shielding and MAGLEV", *American Journal of Physics*, **60** 693-711.

The calculation in section 3 is taken almost completely from

Y. Levin, F. L. da Silveira and F. B. Rizzato, "Electromagnetic braking: a simple quantitative model", *American Journal of Physics*, **74** 815-817.