AP Physics B. Solving the Spring Equation Numerically.

A mass m attached to a spring with constant k satisfies the equation

$$F_{\text{net}} = ma = -kx$$

or, solving for a,

$$a = -\frac{k}{m}x$$

The spring is the first (and nearly the only) force we will look at this year that is *not* a constant. When the net force on an object is a constant, Newton's Second Law says that the acceleration also will be constant, and Galileo's equations will tell us the position and velocity of the object. How can we predict the object's position if, as with a mass on a spring, the acceleration changes from moment to moment?

This turns out to be, in general, a very tough question to solve. The appropriate tool for the job is calculus, in particular a branch devoted to *differential equations*. There are undergraduate and graduate level courses devoted entirely to the study of these things; some are easy, and some are extremely difficult.

Fortunately, the calculus for this particular system is very easy, and we will see how to do it in class. There are, however, situations that calculus cannot handle easily, and in those circumstances an approximate method is used. The purpose of this handout is to show you how the approximate method works, even though we will also handle this problem exactly with calculus in a day or two.

The only reason Galileo's equations don't work for the spring is because the acceleration changes as time goes by. Get around this problem by considering only small intervals of time, during which the acceleration is nearly constant. Recall Galileo I:

$$v = v_o + at$$

and Galileo III:

$$x - x_o = \tfrac{1}{2}(v + v_o)t$$

Rewrite these as follows:

$$a_{\text{avg}} = \frac{v_{\text{later}} - v_{\text{earlier}}}{\Delta t}$$

or

$$v_{\text{later}} = v_{\text{earlier}} + a_{\text{avg}}\Delta t$$

and

$$x_{\text{later}} = x_{\text{earlier}} + v_{\text{avg}}\Delta t$$

If Δt is very small, perhaps these equations are correct even when the acceleration changes. it might appear that we can't learn very much if Δt isn't very large. Say we know the values of x, v and a at some time t. We can only determine the values of x, v and a a brief moment later. But nothing prevents us from repeating the process, over and over again, to determine these values at any time at all. Years ago this would have been very tedious, but now there are computer programs (in particular, spreadsheets like Microsoft *Excel*) to do this work for us. Let's have a look at how this goes.

We need some notation. Let

$$t_n = n\Delta t$$

where the size of Δt is to be determined. In exactly the same way,

$$x_n = x(t_n)$$

and similarly with a and v. Let's say for our spring that $k/m = 4\ \text{Hz}^2$. (It doesn't matter what this ratio equals, but we need *some* number to work with.) Then because of Newton's Second Law,

$$a_n = -4x_n \tag{1}$$

The equivalent of Galileo III is

$$x_{n+1} = x_n + \tfrac{1}{2}\left(v_{n+1} + v_n\right)\Delta t$$

Of course, for this to work, we'd have to have v_{n+1} in hand. How are we going to get that? Presumably from a_{n+1}. Where are we going to get *that*? Well, from Newton's Second Law. But wait a minute. We cannot get a_{n+1} except from x_{n+1}; and that's what we're looking for in the first place! So we have to be a little craftier: Substitute $v_{n+\frac{1}{2}}$ for $\frac{1}{2}(v_{n+1}+v_n)$. That is, our Galileo III equation is

$$x_{n+1} = x_n + v_{n+\frac{1}{2}}\Delta t \tag{2}$$

How do we get $v_{n+\frac{1}{2}}$? Easy. Galileo I says

$$v_{\text{later}} = v_{\text{earlier}} + a_{\text{avg}}\Delta t$$

which we can rewrite as

$$v_{n+\frac{1}{2}} = v_{n-\frac{1}{2}} + a_n\Delta t \tag{3}$$

So, if we have earlier values of x, a and v, we can get later values, up to any time we like. There is one minor complication at first, but after that, it's really easy.

Begin with a choice of Δt, say $\Delta t = 0.1$ second. (For notational convenience, I'm going to suppress all the units.) We have to know the initial values of x and v. Choose $x_o = 0.1$, and $v_o = 0$. From Newton's Second Law, with the choice of $k/m = 4$, we have

$$a_o = -4x_o = -0.4$$

The next value we are going to calculate is $v_{\frac{1}{2}}$. According to (3) above,

$$v_{\frac{1}{2}} = v_{-\frac{1}{2}} + a_o\Delta t$$

The problem is that we have no idea what $v_{-\frac{1}{2}}$ is; the earliest value of v we have is v_o. Consequently we have to use a special starting value for $v_{\frac{1}{2}}$. Try

$$v_{\frac{1}{2}} = v_o + a_o \times \tfrac{1}{2}\Delta t = 0 + (-0.4) \times \tfrac{1}{2}0.1 = -0.02$$

(Note that the time interval between these values of v is $\frac{1}{2}\Delta t$.) Using our rewritten Galileo III gives the first calculation of x;

$$x_1 = x_o + v_{\frac{1}{2}} \times \Delta t = 0.1 + (-0.02) \times 0.1 = 0.098$$

Newton's Second Law then gives (see (1) above)

$$a_1 = -4x_1 = -4 \times 0.098 = -0.392$$

which allows us to find $v_{3/2}$ and x_2, which provides a_2, which allows us to find $v_{5/2}$, x_3, a_3, and so on. This is where *Excel* comes in. The spreadsheet can be used to do all the grunt work for us. Here are the next two steps by hand:

$$v_{3/2} = v_{1/2} + a_1\Delta t = -0.02 + (-0.392) \times 0.1 = -0.0592$$
$$x_2 = x_1 + v_{3/2}\Delta t = 0.098 + (-0.0592) \times 0.1 = 0.09208$$
$$a_2 = -4x_2 = -0.3683$$

$$v_{5/2} = v_{3/2} + a_2\Delta t = -0.0592 + (-0.3683) \times 0.1 = -0.0960$$
$$x_3 = x_2 + v_{5/2}\Delta t = 0.09208 + (-0.0960) \times 0.1 = 0.0825$$
$$a_3 = -4x_3 = -0.3299$$

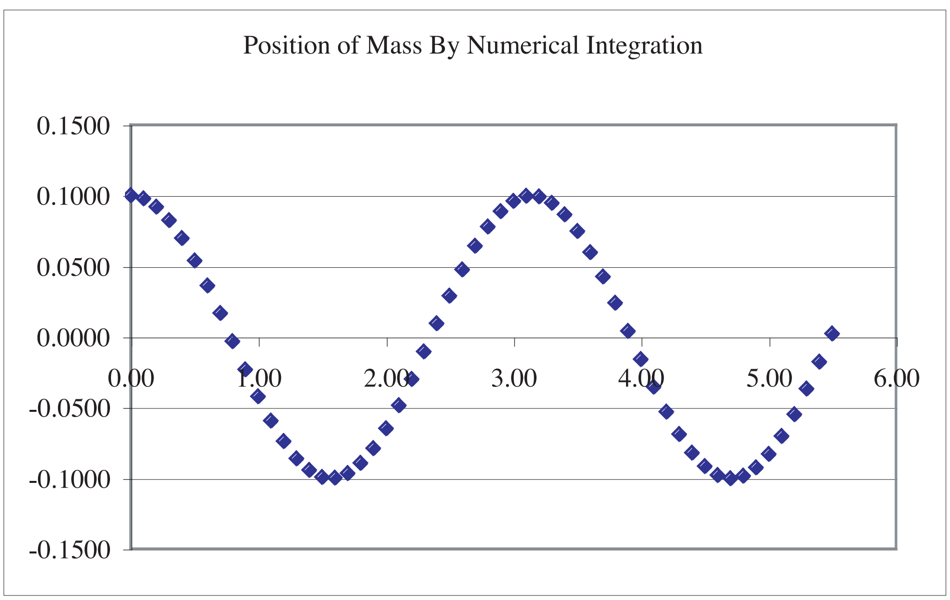
And here are the first few steps of the same calculation done by *Excel*:

t	x	v	a
0.00	0.1000	0.0000	-0.4000
0.05	—	-0.0200	—
0.10	0.0980	-0.0592	-0.3920
0.20	0.0921	-0.0960	-0.3683
0.30	0.0825	-0.1290	-0.3299
0.40	0.0696	-0.1569	-0.2783
0.50	0.0539	-0.1784	-0.2156
0.60	0.0360	-0.1928	-0.1442
...	...	...	...

If we refine our time step to a smaller value, e.g. 0.05, and let the calculation go for many steps, it is easy to determine the period of the oscillation:

t	x	v	a
...	...	...	...
3.10	0.0997	0.0061	-0.3987
3.15	0.1000	-0.0139	-0.3999
3.20	0.0993	-0.0337	-0.3971
3.25	0.0976	-0.0533	-0.3904
...	...	...	...

Clearly, the mass returns to its original position $x = 0.1$ when $t = 3.15$, not terribly different from π (the actual value.) *Excel* can also graph the motion (for some reason, Microsoft calls this a "chart"):



The numerical integration can tell us about the motion of the mass on the spring, even if we don't get from it a nice little equation, as we might get from calculus. You should know that some motions are so complicated that even calculus fails to provide a simple formula for them; in such cases numerical methods are just about the only way to study them.

References. Richard P. Feynman et. al., *The Feynman Lectures on Physics*, vol. 1, Chapter 9, pp 9-5 through 9-9. The best book on numerical methods in physics is W. H. Press, B. Flannery, S. A. Teukolsky, and W. Vetterling, *Numerical Recipes*, Cambridge U. Press; a cheap alternative is A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, Dover Publications.

v.1.0 10 Dec 2005