Phys 300/301
Physics: Algebra/Trig
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10.0 Elasticity & Oscillations

Elasticity is the ability of a system that was once distorted to spontaneously return to its original configuration. In going through this motion the distorted object will frequently oscillate through equilibrium before returning to its original state. All things vibrate and oscillate so this is a general introduction to the motion.

10.1 Elasticity & Hooke's Law

Robert Hooke was one of the first to attempt to describe the behavior of materials under the influence of applied forces. Instead of assuming that all solid bodies remained rigid until fracture or failure, he analyzed what occurred to the internal structure when a force was applied.

Any object that will return to its original state after a period of distortion is referred to as elastic. Objects such as chewing gum, clay, and wax do not return to their original configurations; these objects are under the category of a plastic. Hooke's Law states that elastic materials deform in proportion to the load they support. It applies linearly only – objects like bars, rods, wires, springs, diving board, etc. These objects, when a force is applied, tend to contract or extend linearly by a distance proportional to the force.

[linearly elastic] $$F = kx \tag{10.1}$$

The value of k is the spring/elasticity constant that is a measure of the stiffness of the object being deformed. It has units of N/m. The value x, sometimes given as s or d, is the compression or extension length. Generally, the compression or extension distance is given in the same direction of the force. If force is positive, x is positive, and vice versa.

As expected, when a spring is stretch, it contains a given amount of potential energy. This type of potential energy is known as elastic potential energy. The work done against the force on a spring is the area under the force-distance curve, which equals the change in potential energy of the system.

$$\Delta PE = \tfrac{1}{2}\, x\,(kx)$$

$$\Delta PE = \tfrac{1}{2}\, kx^2 \tag{10.2}$$

Harmonic Motion

Any vibrating system that oscillates in periodic motion is said to be undergoing simple harmonic motion. A single sequence of this motion is known as a cycle. The time it takes for the completion of one full cycle is known as a period (T). For example, the period of one Earth rotation is approximately 24 hours.

The period is the number of units of time per cycle; the number of cycles per unit of time is known as the frequency (f). Frequency is simply the inverse of the period. The unit for frequency is the hertz (Hz).

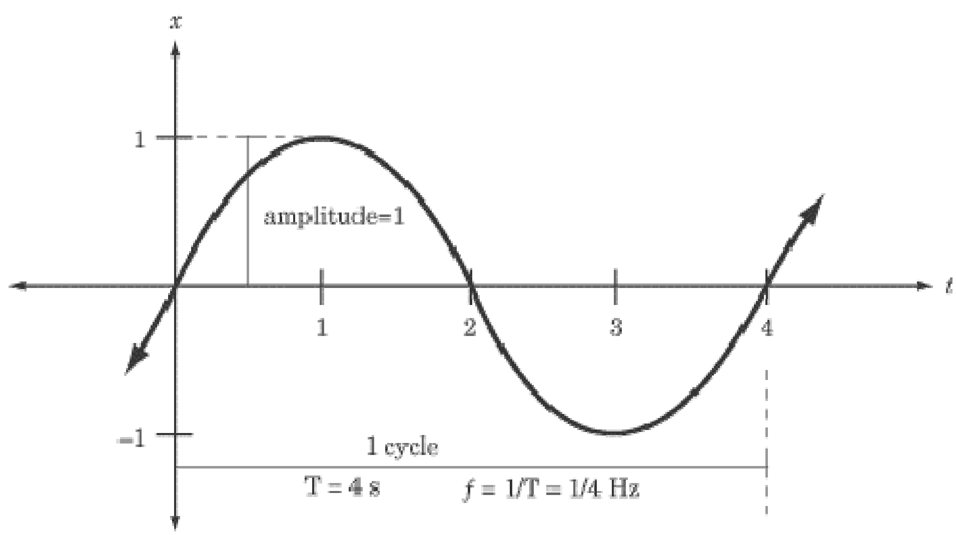
$$f = \frac{1}{T} \qquad (10.3)$$

If an object in circular orbit at a constant rate we say it has an angular speed, ω (omega). Each time the object completes an orbit, it sweeps through an angle of 2π radians. If the number of cycles the object goes through is *f*, the number of radians it moves through per second is 2π*f*. Thus we have angular speed, or angular frequency.

$$\omega = 2\pi f \qquad (10.4.1)$$

$$\omega = \frac{2\pi}{T} \qquad (10.4.2)$$

Amplitude is the third major part of oscillations and waves. Frequency (*f*), period (T), and amplitude (A) and connected through waves as indicated in the following graph:



As you can see, the period of the graph is the time it takes for a wave to be completed, the frequency is the inverse of the period, and the amplitude is the maximum height of the wave.

10.5 Simple Harmonic Motion

Realistically, objects that oscillate do so in a very complicated fashion. For the purposes of this course, we will focus on Simple Harmonic Motion (SHM). That means that motion will be sinusoidal with a single frequency. There are many natural objects that vibrate with dominant SHM.

An object undergoing simple harmonic motion can be located with respect to the origin. Consider motion around a circle. At any point in time an object can be located with the angle θ.

Since the linear speed is constant, the corresponding angular speed of the object is constant, so the angle θ can be determined by:

$$\theta = \omega t \tag{10.5}$$

The projection of the position vector when the object is at an angle θ is the distance the object is away from the origin. This displacement is given by the following:

$$x = A\cos\theta \tag{10.6.1}$$

$$x = x_{\max}\cos\omega t \tag{10.6.2}$$

The displacement oscillates from the maximum value of x and the minimum value of x, just as a cosine wave oscillates from +π to -π. The maximum displacement of x is known as the amplitude of the oscillation. Mathematically, one can see that the maximum value of x, or the amplitude (A) occurs when θ = 0 (cos θ = 1).

As the displacement changes with time, velocity and acceleration are as well. As an object moves around a circular path of radius r (r = A) with a constant linear speed v = Aω, its projection remains directly below it on the x-axis. Initially, the velocity vector is pointing in the negative x direction. The projection of the velocity vector is then given by:

$$v_x = -A\omega\sin\omega t \tag{10.7}$$

Because the angular velocity depends on the frequency, the speed of wave itself depends directly on the frequency of the waves. A point on the wave cannot oscillate at a high frequency without having a high speed. With some quick calculus finding the acceleration is simple, for our purposes, we will not go through the derivation.

$$a_x = -A\omega^2\cos\omega t \tag{10.8}$$

The oscillator has its maximum acceleration at the limits of its motion when the velocity is zero. The acceleration of the simple harmonic oscillator is proportional to its displacement.

10.6 Elastic Restoring Force

When a system oscillates naturally, without being driven by an external energy source, it does so by moving against a restoring force. A quantity of potential energy is given to the system at first, then the system oscillates around the equilibrium position due to the restoring force. The system's energy changes from potential to kinetic in an infinite cycle, assuming there is no loss of energy to friction or heat.

An oscillating spring most resembles simple harmonic motion (SHM). Recalling Hooke's Law, $F = kx$ the force that is exerted on an object to make it distort (stretch, compress, twist, bend, etc) is related to the displacement from equilibrium that the object undergoes. From Newton's Third Law, the force exerted by the spring is $F = -kx$. Because force and acceleration are proportional, it is legitimate to say that the acceleration of a body is proportional to the opposite of its displacement from equilibrium.

Mass on a Spring

The most commonly used example of a simple harmonic oscillator is a mass on a spring. When the mass on the spring is initially displaced some distance x and released from rest, that displacement is the maximum attainable value and the mass will oscillate between ±x. If we compare the mass on a string's oscillation with the acceleration of a body undergoing simple harmonic motion, we can calculate the frequency at which the system will oscillate by itself.

[mass on a spring] $$a = -(\frac{k}{m})x$$

[SHM] $$a = -\omega^2 x$$

[angular frequency] $$\omega = \sqrt{\frac{k}{m}} \qquad (10.9)$$

[linear frequency] $$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (10.10)$$

[period] $$T = 2\pi\sqrt{\frac{m}{k}} \qquad (10.11)$$

The stiffer the spring is, the larger the elastic force constant k, and thus a higher vibration frequency and shorter period. Objects such as solid iron bars have a very large value of k, so high you cannot see the oscillations when a mass it attached to it.

10.7 The Pendulum

Galileo performed a series of experiments and calculated that **the period of the pendulum is independent of the mass and determined by the square root of its length**. Considering what we know about how the acceleration due to gravity is constant on all masses, this discovery should come as no surprise, in present day.

When the pendulum swings at angles, tangential acceleration can be calculated much like the way it is calculated on a ramp.

$$a = -g\sin\theta$$

At smaller angles, sin θ = l / L, where L is the length of the pendulum and l is the change in position of the mass at the bottom of the pendulum.

$$a \approx -(\frac{g}{L})l$$

It follows from the solution to the acceleration of a simple harmonic oscillator.

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{10.12}$$

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{10.13}$$

At any location, the period of the pendulum is dependent only on the square root of its length. It is important to note that mass has nothing to do with the period of a pendulum. This is as expected since the objects acceleration is due to gravity, and gravity acts on all masses equally.